

# ON MULTIPLY TWISTED KNOTS THAT ARE SEIFERT FIBERED OR TOROIDAL

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**ABSTRACT.** We consider knots whose diagrams have a high amount of twisting of multiple strands. By encircling twists on multiple strands with unknotted curves, we obtain a link called a generalized augmented link. Dehn filling this link gives the original knot. We classify those generalized augmented links that are Seifert fibered, and give a torus decomposition for those that are toroidal. In particular, we find that each component of the torus decomposition is either “trivial”, in some sense, or homeomorphic to the complement of a generalized augmented link. We show this structure persists under high Dehn filling, giving results on the torus decomposition of knots with generalized twist regions and a high amount of twisting. As an application, we give lower bounds on the Gromov norms of these knot complements and of generalized augmented links.

## 1. INTRODUCTION

This paper continues a program to understand the geometry of knot and link complements in  $S^3$ , given only a diagram of the knot or link. Each knot complement decomposes uniquely along incompressible tori into hyperbolic and Seifert fibered pieces, by work of Jaco–Shalen [15] and Johannson [16]. By Mostow–Prasad rigidity, the metric on the hyperbolic pieces is unique. Thus this geometric information on the complement is completely determined by a diagram of the knot. However, reading geometric information off of a diagram seems to be difficult.

In recent years, techniques have been developed to relate geometric properties to a diagram for classes of knots and links admitting particular types of diagrams, such as alternating [18], and highly twisted knots and links [22, 21, 12]. However, many links of interest to knot theorists and hyperbolic geometers do not admit these types of diagrams. These include Berge knots [6, 4, 5], twisted torus knots and Lorenz knots [7], which contain many of the smallest volume hyperbolic knots [9]. These knots admit diagrams that are highly non-alternating, that have few twists per twist region, but contain regions where multiple strands of the diagram twist around each other some number of times. The ideas of this paper and a companion paper [20] grew out of a desire to understand geometric properties of these “multiply twisted” knots and links, given only a diagram. The results here give a first step towards such an understanding.

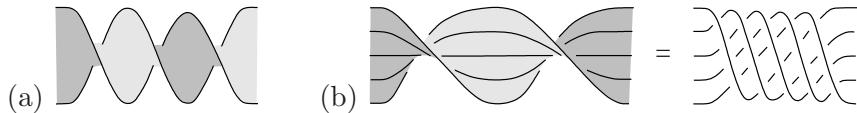


FIGURE 1. (a) A twist region. (b) A generalized twist region. Multiple strands lie on the twisted ribbon surface.

In this paper, we consider a class of multiply twisted knots in  $S^3$ , described below, and classify those which are *not* hyperbolic. We completely determine those which are Seifert fibered, and describe the unique torus decomposition (the JSJ decomposition) for those which are toroidal. We obtain our results by *augmenting* diagrams of the multiply twisted knots, that is, encircling regions of the diagram where multiple strands twist about each other by a simple closed curve, called a *crossing circle*. This generalizes a construction of Adams [1]. Since all knots are obtained by Dehn filling some generalized augmented link, the geometric properties of these links are also interesting. In [20], we describe geometric properties of those generalized augmented links which are known to be hyperbolic. Combining these results leads to the JSJ decomposition of certain multiply twisted knots.

To state our results carefully, we need some definitions.

**1.1. Generalized augmented links.** First, although our main results relate to knots in  $S^3$ , in fact many results in this paper apply more generally to knots and links in a 3-manifold  $M$ . We will assume  $M$  is compact, with (possibly empty) boundary consisting of tori, and  $M$  admits an orientation reversing involution fixing a surface  $S$ . For example,  $S^3$  is such a manifold, taking  $S$  to be a separating 2-sphere. A solid torus  $V$  is another example, with  $S$  a Möbius band or annulus.

Let  $K$  be a knot or link in  $M$  that can be ambient isotoped into a neighborhood of  $S$ . We define a *diagram* of the knot or link  $K$  with respect to the surface  $S$  to be a projection of  $K$  to  $S$  yielding a 4-valent graph on  $S$  with over-under crossing information at each vertex.

Given a diagram, we may define twisting, twist regions, and generalized twist regions exactly as in [20], whether or not our link is in  $S^3$ . We review the definitions briefly here. More precise statements are found in [20].

A *twist region* of a diagram is a region in which two strands twist about each other maximally, as in Figure 1(a). Note that the two strands bound a “ribbon surface” between them. A *generalized twist region* is a region of a diagram in which multiple strands twist about each other maximally, as in Figure 1(b). Note that all strands lie on the ribbon surface bounded between the outermost strands. A *half-twist* of a generalized twist region consists of a single crossing of the two outermost strands, which flips the ribbon surface over once. Figure 1(b) shows a single full-twist, or two half-twists of five strands.

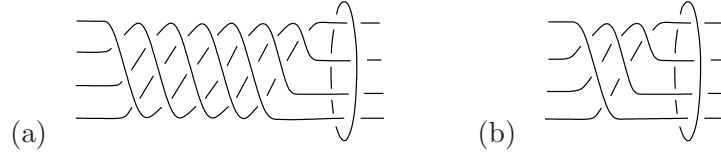


FIGURE 2. (a) Encircle each twist region with a crossing circle. (b) Link  $L$  given by removing full-twists from the diagram.

We may group all crossings of a diagram into generalized twist regions, so that each crossing is contained in exactly one generalized twist region. This is called a *maximal twist region selection*, and is not necessarily unique. For example, in Figure 1(b), we could group all crossings into one generalized twist region on five strands, or group each crossing into its own twist region on two strands.

Given a maximal twist region selection, at each generalized twist region insert a *crossing circle*, i.e. a simple closed curve  $C_i$  encircling the strands of the generalized twist region, bounding a disk  $D_i$  perpendicular to the projection plane of the diagram. The complement of the resulting link is homeomorphic to the complement of the link whose diagram is obtained by removing all full-twists from each generalized twist region. This is illustrated in Figure 2. The resulting link, with crossing circles added and all full-twists removed, is defined to be a *generalized augmented link*. We will always assume such a link contains at least one crossing circle, to avoid trivial cases.

Note that if  $L$  is a generalized augmented link, obtained by augmenting a knot  $K$  in  $M$ , then  $M - K$  is obtained from  $M - L$  by Dehn filling. Let  $\mathcal{N}(C_i)$  denote a small embedded tubular neighborhood of  $C_i$  in  $M$ . Let  $\mu_i$  be (the isotopy class of) the meridian of  $\mathcal{N}(C_i)$  (i.e.  $\mu_i$  bounds a disk in  $M$ ), and let  $\lambda_i = \partial D_i$  be the longitude. Suppose  $n_i$  full-twists were removed at  $C_i$  to go from the diagram of  $K \cup (\cup C_j)$  to that of  $L$ . Then Dehn filling along the slope  $\mu_i + n_i \lambda_i$  on  $\partial \mathcal{N}(C_i)$ , for each  $i$ , yields  $M - K$ . See, for example, Rolfsen [25] for a more complete description of this process.

We refer to this type of Dehn filling as *twisting* along the disk  $D_i$ , or along  $C_i$ , or, when  $D_i$  or  $C_i$  are understood, simply as *twisting*. Note any link in  $S^3$  is obtained by twisting some generalized augmented link.

Finally, we wish to use diagrams of knots that do not involve unnecessary twisting. That is, we wish them to be *reduced* in the sense of the following definition, which we will use to generalize Lackenby's definition of twist reduced [18], and Menasco–Thistlethwaite's definition of standard [19].

**Definition 1.1.** A generalized augmented link with knot strands  $K_j$  and crossing circles  $C_i$  is said to be *reduced* if the following hold:

- (1) (Minimality of twisting disks) The twisting disk  $D_i$  with boundary  $C_i$  intersects  $\cup K_j$  in  $m_i$  points, where  $m_i \geq 2$ , and  $m_i$  is minimal

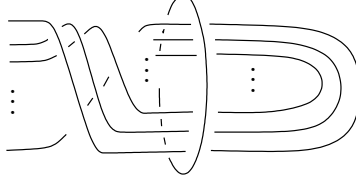


FIGURE 3. Untwist at a homotopically trivial crossing circle.

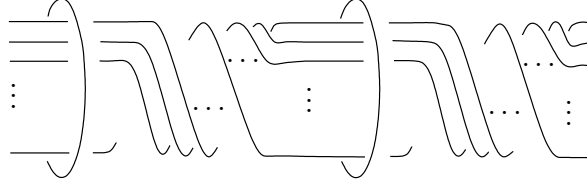


FIGURE 4. Concatenate twists when crossing circles bound an annulus.

over all disks in  $M$  bounded by  $C_i$ . That is, if  $E_i$  is another disk embedded in  $M$  with boundary  $C_i$ , disjoint from all other crossing circles, then  $|E_i \cap (\cup K_j)| \geq m_i$ .

- (2) (No redundant twisting) There is no annulus embedded in  $M - L$  with one boundary component isotopic to  $\partial D_j$  on  $\partial \mathcal{N}(C_j)$  and the other isotopic to  $\partial D_i$  on  $\partial \mathcal{N}(C_i)$ ,  $i \neq j$ .
- (3) (No trivial twisting) There is no annulus embedded in  $M - L$  with one boundary component isotopic to  $\partial D_j$  on  $\partial \mathcal{N}(C_j)$  and the other boundary component on  $\partial M$ .

These conditions allow us to rule out unnecessary crossing circles and generalized twist regions. For example, condition (1) prohibits “nugatory” twist regions, such as those shown in Figure 3. Condition (2) rules out redundant generalized twist regions and their associated crossing circles, such as shown in Figure 4, where two crossing circles encircle the same generalized twist region.

**1.2. Results.** We are now ready to state the main results of this paper.

In Section 3, we classify all reduced augmented links which are Seifert fibered. This is the content of Theorem 3.8. As a consequence, we obtain the following result.

**Corollary 3.9.** *Let  $K$  be a knot in  $S^3$  which has a diagram  $D$  whose augmentation is a Seifert fibered reduced augmented link. Then  $K$  is a  $(2, q)$  torus knot.*

In Section 4, we describe a torus decomposition of generalized augmented links. In particular, in Theorem 4.1, we show that components of such a torus decomposition are either in some sense trivial, or atoroidal reduced

augmented links. Applying this to links in  $S^3$ , using results on hyperbolic generalized augmented links of [20], we obtain a torus decomposition for all links in  $S^3$  obtained by high Dehn fillings of toroidal generalized augmented links. This is Theorem 4.7. In particular, if at least 6 half-twists are inserted when we twist along  $C_i$ , then the torus decomposition of  $M - K$  will agree with that of  $M - L$ , aside from “trivial” pieces.

We wish to apply these results to as many knots in  $S^3$  as possible. In Section 5, we show that any knot in  $S^3$  admits a diagram whose augmentation is reduced. Thus the above results will apply at least to their augmentations. This is Theorem 5.1.

In Section 6, we apply these results to knots in  $S^3$ , to determine the JSJ decomposition of multiply twisted knots. Our main result is the following.

**Theorem 6.4.** *Let  $K$  be a knot in  $S^3$  which is toroidal, with a twist-reduced diagram and a maximal twist region selection with at least 6 half-twists in each generalized twist region. Let  $L$  denote the corresponding augmentation. Then there exists a sublink  $\hat{L}$  of  $L$ , possibly containing fewer crossing circles, such that:*

- (1) *The essential tori of the JSJ decomposition of  $S^3 - K$  are in one-to-one correspondence with those of  $S^3 - \hat{L}$ .*
- (2) *Corresponding components of the torus decompositions have the same geometric type, i.e. are hyperbolic or Seifert fibered.*
- (3) *Essential tori of  $S^3 - \hat{L}$  and  $S^3 - K$  form a collection of nested tori, each bounding a solid torus in  $S^3$  which contains  $K$ , and is fixed under a reflection of  $S^3 - L$ .*

Putting this theorem with the results on hyperbolic geometry of generalized augmented links in [20], we obtain as an application a lower bound on the Gromov norm of such knots.

**Theorem 7.3.** *Let  $K$  be a knot in  $S^3$  which is toroidal, with a twist-reduced diagram at least 7 half-twists in each generalized twist region. Let  $L$  denote the corresponding augmentation, and let  $\hat{L}$  denote the sublink of Theorem 6.4. Let  $t$  denote the number of crossing circles of  $\hat{L}$ . Then the Gromov norm of  $S^3 - K$  satisfies*

$$\|[S^3 - K]\| \geq 0.65721(t - 1).$$

**1.3. Comments and additional questions.** The results of this paper give geometric information based purely on diagrammatical properties of extensive classes of knots. However, because of the high amount of twisting required, for example in Theorem 6.4, these classes still do not include examples of many knots. Considering knots which are not included leads to two interesting remaining questions.

First, can the results of Theorem 6.4 be sharpened to require fewer half-twists? We can construct examples of atoroidal knots  $K$  whose geometric type (hyperbolic or Seifert fibered) does not agree with that of the corresponding reduced augmented link  $L$ . However, in all these examples, at

least one generalized twist region contains fewer than 3 half-twists. In [3], Ait-Nouh, Matignon, and Motegi, working on a related question, show that when exactly one crossing circle is inserted into the diagram of an unknot, and then the unknot is twisted, inserting at least 4 half-twists, the geometric type of the resulting knot (Seifert fibered, toroidal, or hyperbolic) agrees with that of the unknot union the crossing circle. While these results do not apply to generalized augmented links, the result requiring only 4 half-twists is intriguing.

Secondly, given an arbitrary diagram of a knot, is there a way to optimize the maximal twist region selection? There are many ways to choose a maximal twist region selection. For example, Figure 1(b) could either be seen as a single full-twist of 5 strands, or as 20 half-twists, each of 2 strands. To apply the results of this paper, it seems we would want to select generalized twist regions to maximize the number of half-twists in each twist region. Is there an algorithm that, given a diagram of a knot  $K$ , produces a reduced diagram and a maximal twist region selection with the highest number of half-twists per generalized twist region possible for  $K$ ? Results along these lines would be interesting.

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## 2. REFLECTION

A generalized augmented link in a manifold  $M$  admits a reflection through a surface fixed pointwise, just as in the case of links in  $S^3$  [23, Proposition 3.1]. This reflection is necessary for many of the results that follow, and so we state it first.

**Proposition 2.1.** *Let  $M$  be a 3-manifold with torus boundary which admits an orientation reversing involution fixing a surface  $S$  pointwise. Let  $K$  be a link in  $M$  which may be isotoped to lie in a neighborhood of  $S$ . Finally, let  $L$  be an augmentation of a diagram of  $K$ , with crossing circles  $\{C_1, \dots, C_n\}$  and knot strands  $\{K_1, \dots, K_m\}$ . Then  $M - (\cup C_i)$  admits an orientation reversing involution  $\sigma$  which fixes a surface  $P$  pointwise, and each  $K_j$  is embedded in  $P$ . In particular,  $M - L$  admits an orientation reversing involution  $\sigma$ .*

*Proof.* Isotope crossing circles to be orthogonal to  $S$ , preserved by the reflection of  $M$  through  $S$ . If there are no half-twists in the diagram of  $L$ , then all components  $K_j$  of  $L$  are embedded in  $S$ . Hence the reflection in  $S$  preserves each  $K_j$  as well as each  $C_j$ , so  $P = S$  and the involution is the restriction of the involution of  $M$  to  $M - L$ .

If there are half-twists in the diagram of  $L$ , then the reflection of  $M$  through the surface  $S$  gives a new link  $L'$  in which all the directions of the crossings at each half-twist have been reversed. Let  $\tau$  be the homeomorphism of  $M - (\cup C_j)$  which twists exactly one full time in the opposite

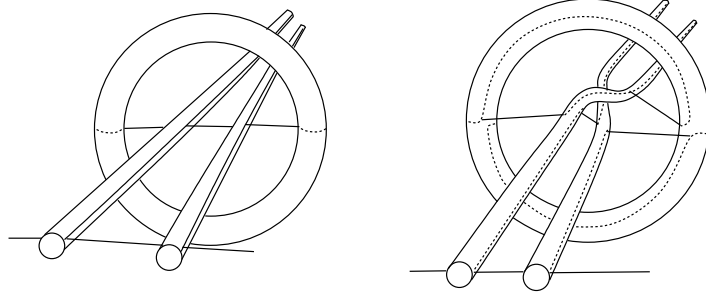


FIGURE 5. Left:  $P \cap \partial\mathcal{N}(C_i)$  consists of two meridians when there is no half-twist. Right: Under a half-twist,  $P \cap \partial\mathcal{N}(C_i)$  has boundary shown by the dotted lines.

direction of these half-twists at each corresponding crossing circle. Applying  $\tau$  changes the diagram to one in which the crossings of half-twists have been reversed again, hence to the diagram of  $L$ . So  $M - L'$  is homeomorphic to  $M - L$ , and the orientation reversing involution  $\sigma$  of  $M - (\cup C_j)$  is given by reflection of  $M$  in  $S$  followed by the homeomorphism  $\tau$ .

Finally, we describe the surface  $P$  fixed pointwise by  $\sigma$  in the case of half-twists. In this case,  $P$  is equal to  $S$  outside a neighborhood of those crossing disks for which the corresponding crossing circle  $C_i$  bounds a half-twist. Inside such a neighborhood, the surface  $P$  follows the ribbon surface of the half-twist between the outermost knot strands. Between  $C_i$  and the outermost knot strands,  $P$  runs over the overcrossing, under the undercrossing, and meets up with the surface  $S$  on the opposite side of the link. Its boundary  $P \cap \partial\mathcal{N}(C_i)$  runs twice along the meridian of  $\partial\mathcal{N}(C_i)$ , once along the longitude, as in Figure 5.  $\square$

In [20, Lemma 3.1], we showed that in this setting, the slopes  $\partial D_i$  on  $\partial\mathcal{N}(C_i)$  will meet the surface  $P$  of Proposition 2.1 exactly twice. We will use this fact here, and so we state it as a lemma.

**Lemma 2.2.** *Let  $L$  be a generalized augmented link in  $M$ , with reflective surface  $P$  from Proposition 2.1 and twisting disks  $D_i$ . Then for each  $i$ ,  $\partial D_i$  meets  $P$  exactly twice on  $\partial\mathcal{N}(C_i)$ .*  $\square$

### 3. THE SEIFERT FIBERED CASE

In this section, we classify Seifert fibered augmented links.

**3.1. Incompressibility of surfaces.** Let  $M - L$  be the complement of a generalized augmented link. By Proposition 2.1,  $M - L$  admits an involution  $\sigma$  which fixes a surface  $P$ . We show that  $P$  and the (punctured) twisting disks  $D_i - (D_i \cap L) \subset M - L$  are incompressible.



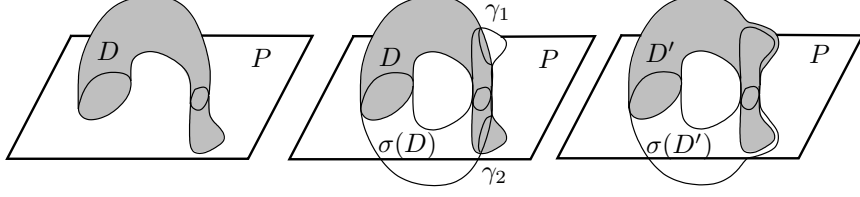


FIGURE 6. Right:  $D$  (shaded). Middle:  $D$  and  $\sigma(D)$  intersect at  $\gamma_1$  and  $\gamma_2$ . Left: Replace the interior of  $\gamma_1$  and push off to reduce the number of intersections by 2.

**Lemma 3.1.** *Let  $N$  be an irreducible 3-manifold with torus boundary components which admits an orientation reversing involution  $\sigma$  with fixes a surface  $P$ . Then  $P$  is incompressible.*

*Proof.* Suppose not. Suppose  $D$  is a compressing disk for  $P$ . Then  $\partial D$  lies on  $P$ , so is fixed by  $\sigma$ , and  $\sigma(D)$  is a disk whose boundary agrees with that of  $D$ . We first show we can assume  $D$  and  $\sigma(D)$  are disjoint except on their boundaries. If not, consider the intersections of  $D$  and  $\sigma(D)$ . Note these intersections consist of closed curves on  $D$  and on  $\sigma(D)$  — not arcs, since  $\partial D$  is contained on  $P$ , and  $\sigma$  acts as a reflection in a small neighborhood of  $P$ .

Let  $\gamma_1$  be a circle of intersection of  $D$  and  $\sigma(D)$  which is innermost on  $\sigma(D)$ . Then  $\gamma_1$  lies on  $D$  and  $\sigma(D)$ , hence  $\gamma_2 = \sigma(\gamma_1)$  lies on  $\sigma(D)$  and  $D$ , and  $\gamma_2$  is innermost on  $D$ . Surger: Replace  $D$  by replacing the disk bounded by  $\gamma_1$  on  $D$  with the disk bounded by  $\gamma_1$  on  $\sigma(D)$ , and push off  $\sigma(D)$  slightly. Call this new disk  $D'$ .

We claim that the number of intersections  $|D' \cap \sigma(D')|$  is now less than  $|D \cap \sigma(D)|$ . Outside a neighborhood of the disk bounded by  $\gamma_1$ ,  $D'$  agrees with  $D$ . Hence outside a neighborhood of the disk bounded by  $\gamma_2 = \sigma(\gamma_1)$ ,  $\sigma(D')$  agrees with  $\sigma(D)$ .

There are two cases to consider. First, if  $\gamma_2$  is outside the disk bounded by  $\gamma_1$  on  $D$ , then  $\gamma_2$  and the disk it bounds in  $D$  are still contained in  $D'$  (since  $D'$  agrees with  $D$  outside  $\gamma_1$ ). Similarly,  $\gamma_1$  will be outside the disk bounded by  $\gamma_2$  on  $\sigma(D)$ , so  $\gamma_1$  and the disk bounded by  $\gamma_1$  on  $\sigma(D)$  will still remain on  $\sigma(D')$ . An example of this is illustrated in Figure 6.

When we push off  $\sigma(D)$  to form  $D'$ , we may do so equivariantly. So  $D'$  doesn't intersect  $\sigma(D')$  in a neighborhood of the disk bounded by  $\gamma_1$  on  $\sigma(D')$ , and  $\sigma(D')$  doesn't intersect  $D'$  in a neighborhood of the disk bounded by  $\gamma_2$  on  $D'$ . Elsewhere,  $D$  and  $D'$  agree, so other intersections have not changed. Thus the number of intersections has decreased under this operation.

If instead  $\gamma_2$  is inside the disk  $E_1$  bounded by  $\gamma_1$  on  $D$ , then when we surger, even before pushing off,  $\gamma_1$  now bounds a disk  $E$  on  $D'$ , whose image under  $\sigma$  is bounded by  $\gamma_2$  in  $E_1$ , hence is contained in  $E_1$ . But now  $E_1$  is



disjoint from  $D'$ . Since  $D$  and  $D'$  agree elsewhere,  $D'$  and  $\sigma(D')$  have fewer intersections.

Repeating this process a finite number of times, we obtain a compressing disk  $D$  such that  $D$  and  $\sigma(D)$  are disjoint.

Then  $D \cup \sigma(D)$  is a sphere  $S$  in  $N$ . Since  $N$  is irreducible,  $S$  bounds a ball  $B$  in  $N$  whose boundary  $S$  is invariant under  $\sigma$ . Since  $S$  meets  $P$ , so does  $B$ , and hence  $B$  must be preserved by the involution  $\sigma$ .

Now we have an orientation reversing involution of a ball  $B$  which fixes a circle  $\partial D$  on the boundary of the ball and swaps the disks on the boundary. Double the ball across its boundary and extend  $\sigma$ . This gives an orientation reversing involution of  $S^3$  with fixed point set a surface. It follows from work of Smith in the 1930s that the fixed point set must be a 2-sphere containing  $\partial D$ , and  $B$  intersected with this fixed point set must therefore be a disk. But the fixed point set of  $\sigma$  is  $P$ , so  $P$  contains a disk with boundary  $\partial D$ . This contradicts the fact that  $D$  was a compressing disk for  $P$ .  $\square$

We now use this result to show incompressibility of  $D_i - L$ .

**Lemma 3.2.** *Let  $L$  be a reduced generalized augmented link in  $M$  such that  $M - L$  is irreducible, with twisting disks  $D_i$ . Then each  $D_i - L$  is incompressible in  $M - L$ .*

*Proof.* Let  $P$  be the surface of the reflection provided by Proposition 2.1, and suppose, by way of contradiction, that the punctured disk  $D_1 - L$ , say, is compressible. Let  $D$  be a compressing disk in  $M - L$ . Consider  $D \cap P$ . Note  $\partial D \cap P$ , and therefore  $D \cap P$ , cannot be empty, else  $\partial D$  bounds a disk on  $D_1 - L$ . So  $D \cap P$  consists of a non-empty collection of arcs and closed curves on  $D$ .

We may assume  $D \cap P$  does not contain any closed curves on  $D$ , using the incompressibility of  $P$ , Lemma 3.1. For if there is a closed curve component of  $D \cap P$ , it bounds a disk on  $P$ , hence can be isotoped off.

Next we show that we can take  $D$  to be invariant under  $\sigma$ . Consider an outermost arc  $\gamma$  of  $D \cap P$ . This bounds a disk  $E$  whose boundary consists of the arc  $\gamma$  and a portion of  $\partial D$ . Consider  $F = E \cup \sigma(E)$ . This is a disk embedded in  $M - L$  with boundary on  $D_1$ , which is invariant under  $\sigma$ . If  $F$  is a compressing disk for  $D_1 - L$ , then replace  $D$  by  $F$ , and we have a compressing disk invariant under  $\sigma$ . So suppose  $F$  is not a compressing disk for  $D_1 - L$ . Then  $\partial F$  bounds a disk  $E_1$  on  $D_1$ . So  $F \cup E_1$  is a 2-sphere, which must bound a 3-ball by irreducibility. Isotope  $D$  through this ball, pushing  $E$  along  $E_1$  through  $P$ , reducing the number of intersections of  $P$  with  $D$ . Since there are only finitely many intersections of  $D \cap P$ , repeat this process only a finite number of times, and eventually obtain a compressing disk for  $D_1 - L$  invariant under  $\sigma$ , or we reduce to the case  $D \cap P = \emptyset$ , contradicting the fact that  $D$  is a compressing disk.

Now,  $\partial D$  does not bound a disk on  $D_1 - L$ . Hence it must bound a disk  $E_1$  in  $D_1$  punctured by  $L$ . Replace  $D_1$  by replacing  $E_1$  with  $D$ . Then we have a new disk  $D'$  which is invariant under  $\sigma$ , which meets the  $K_j$  fewer

times. But the generalized augmented link  $M - L$  is reduced, which means, by Definition 1.1, that the number of intersections  $D_1 \cap (\cup K_j)$  is minimal over all disks in  $M$  bounded by  $C_1$ . Since  $D'$  is another such disk, this is a contradiction.  $\square$

**3.2. Annuli.** Next, we show a series of results on annuli that are admitted in a generalized augmented link.

**Lemma 3.3** (Lemma 2.5.3 of [10]). *Let  $M$  be an irreducible 3-manifold with torus boundary components, not homeomorphic to  $T^2 \times I$ . Let  $T_1$  and  $T_2$  be boundary components which are incompressible. Suppose  $A_1$  and  $A_2$  are properly embedded annuli in  $M$  with  $\partial A_i = c_{i1} \cup c_{i2}$ , with  $c_{ij}$  on  $T_j$ . Then  $c_{1j}$  is isotopic to  $c_{2j}$  on  $T_j$ ,  $j = 1, 2$ .*

Lemma 3.3 is actually not as general as [10, Lemma 2.5.3]. Because the statement of that lemma is a little different from Lemma 3.3, we reproduce the proof here for convenience.

*Proof.* Assume  $A_1$  and  $A_2$  are in general position. Let  $\Delta_j$ ,  $j = 1, 2$  denote the number of intersections of  $c_{1j}$  and  $c_{2j}$ . Suppose one is nonzero.

There is an isotopy of the  $A_i$  such that  $\Delta_1 = \Delta_2$ , and any arc of intersection of  $A_1 \cap A_2$  runs from one torus to the other. Otherwise, an arc of intersection runs from one torus back to itself. By an innermost arc argument, using the irreducibility of  $M$  and the incompressibility of  $T_i$ , we can isotope  $A_1$  and  $A_2$  to remove this arc of intersection.

Now, we claim we can replace  $A_2$ , if necessary, so that  $c_{1j}$  and  $c_{2j}$  intersect just once. Suppose  $c_{1j}$  and  $c_{2j}$  intersect at least twice. Choose two arcs adjacent to each other on  $A_2$ , say  $a$  and  $a'$ . That is,  $a$  and  $a'$  bound a disk  $E_2$  on  $A_2$  whose interior is disjoint from  $A_1$ . The arcs  $a$  and  $a'$  will also bound a disk  $E_1$  on  $A_1$ , whose interior is not necessarily disjoint from  $A_2$ , but must be disjoint from  $E_2$  by choice of  $E_2$ .

The boundaries of  $E_1$  are  $a$ , an arc on  $c_{11}$  which we denote  $b_1$ ,  $a'$ , and an arc on  $c_{12}$ , which we denote  $d_1$ . Similarly, the boundaries of  $E_2$  are  $a$ ,  $b_2$  on  $c_{21}$ ,  $a'$ , and  $d_2$  on  $c_{22}$ . Then  $E_1 \cup E_2$  gives an annulus  $A$  embedded in  $M$  with boundary components  $b_1 \cup b_2$  on  $T_1$  and  $d_1 \cup d_2$  on  $T_2$ . Since  $\text{int}(b_2 \cap A_1) = \emptyset$  (because  $E_2$  is disjoint from  $A_1$ ),  $A$  has slope  $r_0$ , say, on  $T_1$ , where  $\Delta(r_0, c_{i1}) = 1$ . Hence after isotopy,  $A \cap A_1$  will consist of a single arc (see [10, Figure 2.3]). Replace  $A_2$  with  $A$ .

Now,  $A_1 \cup A_2$  is homeomorphic to  $X \times I$ , where  $X = X \times \{0\}$  is the union of two simple loops on  $T_1$  which intersect transversely in a single point. It has a regular neighborhood homeomorphic to  $N \times I$ , where  $N = N \times \{0\}$  is a regular neighborhood of  $X$  on  $T_1$ . But now  $\partial N$  bounds a disk  $D_1$  on  $T_1$ ,  $\partial N \times \{1\}$  bounds a disk  $D_2$  on  $T_2$ , so  $D_1 \cup \partial N \times I \cup D_2$  is a 2-sphere, which bounds a 3-ball  $B$  in  $M$  since  $M$  is irreducible. Then  $M = N \times I \cup B$  is homeomorphic to  $T \times I$ .  $\square$

**Corollary 3.4.** *Suppose  $M$  is an irreducible 3-manifold with torus boundary components, not homeomorphic to  $T^2 \times I$ , which admits an orientation*

reversing involution  $\sigma$  which fixes a surface  $P$  meeting incompressible components  $T_1$  and  $T_2$  of  $\partial M$ . Suppose  $A$  is an annulus embedded in  $M$  with boundary components lying on  $T_1$  and  $T_2$ . Then the slopes of  $\partial A$  on  $T_i$ ,  $i = 1, 2$ , are preserved by the involution  $\sigma$ .

*Proof.* If  $\sigma$  does not preserve one of the slopes  $(\partial A)_i$  on  $T_i$ , then  $A$  and  $\sigma(A)$  are two distinct annuli embedded in  $M$  with non-isotopic boundary components. This contradicts Lemma 3.3.  $\square$

We now apply these results to generalized augmented links. First, we need to rule out the case that a generalized augmented link might have complement in  $M$  homeomorphic to  $T^2 \times I$ .

**Lemma 3.5.** *Suppose  $L$  is a generalized augmented link in  $M$  and  $M - L$  is irreducible. Then  $M - L$  is not homeomorphic to  $T^2 \times I$ .*

*Proof.* Suppose not. Since  $T^2 \times I$  has just two boundary components, and since we assume any generalized augmented link has at least one crossing circle  $C_1$ , one boundary component of  $M$  corresponds to  $C_1$  and the other to a knot strand. Since the punctured  $D_1$  is incompressible by Lemma 3.2 and 2-sided, it must be either horizontal or vertical in a Seifert fibering of  $T^2 \times I$ . But then  $D_1 - L$  must be an annulus, contradicting the fact that  $m_1 \geq 2$ .  $\square$

Using this fact, we can rule out annuli embedded in  $M - L$ .

**Lemma 3.6.** *Let  $L$  be a generalized augmented link in  $M$  such that  $M - L$  is irreducible. Then there is no annulus embedded in  $M - L$  with one boundary component on  $\partial \mathcal{N}(C_i)$  parallel to  $P \cap \partial \mathcal{N}(C_i)$ , and the other boundary component disjoint from  $\partial \mathcal{N}(C_i)$ .*

*Proof.* Suppose  $A$  is an annulus embedded in  $M - L$  with boundary component  $(\partial A)_i$  on  $\partial \mathcal{N}(C_i)$  parallel to  $P \cap \partial \mathcal{N}(C_i)$ .

Consider the intersection of  $D_i$  with  $A$ . Since  $(\partial A)_i$  is parallel to  $P \cap \partial \mathcal{N}(C_i)$ , and  $\partial D_i$  meets  $P$  twice by Lemma 2.2, we may isotope  $A$  so that  $\partial D_i \cap A$  consists of one or two points, depending on whether  $P \cap \partial \mathcal{N}(C_i)$  consists of two or one components, as illustrated on the left and right of Figure 5, respectively.

If  $A \cap \partial D_i$  consists of one point, then  $P \cap \partial \mathcal{N}(C_i)$  has two meridional components, and  $(\partial A)_i$  is a meridian on  $\partial \mathcal{N}(C_i)$ . But now consider  $A \cap D_i$ . This consists of arcs and curves of intersection. Any arc has two endpoints on  $A \cap \partial D_i$ , so there must be an even number of points of intersection of  $A \cap \partial D_i$ . However, we are assuming there is just one such point. This is a contradiction.

Thus  $A \cap \partial D_i$  must consist of two points, and  $A \cap D_i$  must have a single arc component running from  $\partial D_i$  to  $\partial D_i$ . This arc bounds a disk  $E$  in  $D_i$ , and a disk  $E'$  in  $A$ , since the other boundary component of  $A$  is disjoint from  $\partial \mathcal{N}(C_i)$ . We may assume the interiors of  $E$  and  $E'$  are disjoint by an innermost curve argument, for intersections must be simple closed curves in

both, since  $A \cap D_i$  has just one arc component, and hence if  $E$  and  $E'$  are not disjoint we may isotope them off of each other using irreducibility of  $M - L$ . So  $E \cup E'$  is a disk with boundary on  $\partial\mathcal{N}(C_i)$ . This must bound a disk on  $\partial\mathcal{N}(C_i)$ . By irreducibility of  $M - L$ , we may therefore isotope  $A$  to have no intersections with  $\partial D_i$ , contradicting the fact that  $(\partial A)_i$  is parallel to  $P \cap \partial\mathcal{N}(C_i)$ .  $\square$

**Lemma 3.7.** *If  $L$  is a reduced generalized augmented link in  $M$  such that  $M - L$  is irreducible, then there is no annulus embedded in  $M - L$  with boundary components on  $\partial\mathcal{N}(C_i)$ ,  $\partial\mathcal{N}(C_j)$ , for  $i \neq j$ .*

*Proof.* By Definition 1.1(1), each  $\partial\mathcal{N}(C_i)$  is incompressible. Thus by Corollary 3.4, any embedded annulus  $A$  must have boundary components fixed by  $\sigma$ . At most two slopes on  $\partial\mathcal{N}(C_i)$  are fixed by  $\sigma$ . These are the slopes of  $P \cap \partial\mathcal{N}(C_i)$  and of  $D_i \cap \partial\mathcal{N}(C_i) = \partial D_i$ .

By Lemma 3.6, no boundary component of  $A$  is parallel to  $P \cap \partial\mathcal{N}(C_i)$  or to  $P \cap \partial\mathcal{N}(C_j)$ . By Definition 1.1(2), we cannot have  $(\partial A)_i$  and  $(\partial A)_j$  parallel to  $\partial D_i$  and  $\partial D_j$ . Thus no such annulus exists.  $\square$

**3.3. Seifert fibered augmented links.** We may now classify all Seifert fibered reduced generalized augmented links.

**Theorem 3.8.** *If  $M - L$  is irreducible and Seifert fibered, where  $L$  is a reduced generalized augmented link in  $M$ , then  $L$  has just one crossing circle component  $C_1$ ,  $M - C_1$  is a solid torus,  $P$  is an embedded annulus or Möbius band in  $M - C_1$ , and the knot strands are parallel to the boundary of  $P$ . In particular, if  $P$  is an annulus, there are at least two knot strand components.*

*Proof.* Suppose  $M - L$  is Seifert fibered. First, by Lemma 3.7, there can be no annuli between link components  $C_i$  and  $C_j$ . This implies that there cannot be more than one link component  $C_1$ .

Now,  $D_1 - L$  is incompressible by Lemma 3.2. Since  $D_1 - L$  is 2-sided, it is horizontal or vertical in  $M - L$ . If vertical, it must be an annulus, contradicting the fact that  $m_1 \geq 2$ . So  $D_1 - L$  is horizontal. Then the meridians of the knot strands (i.e. the curves which bound disks in  $M$ ) cannot be Seifert fibers, so the Seifert fibering of  $M - L$  extends to  $M - C_1$ . The base orbifold of  $M - C_1$  is branch covered by the horizontal surface  $D_1$ , hence it is a disk with one singular point. Thus  $M - C_1$  must be a solid torus.

Since  $P$  is incompressible by Lemma 3.1, if it is orientable, then it is an annulus. Because the knot strands are embedded in  $P$  and nontrivial, they must be parallel to the core of the solid torus  $M - C_1$ . In particular, if there is just one knot strand component, then  $M - L$  is homeomorphic to  $T^2 \times I$ , contradicting Lemma 3.5. So in this case there are at least 2 knot strand components.

If  $P$  is non-orientable, then by work of Frohman [11] and Rannard [24],  $P$  is *pseudo-vertical* in a solid torus, meaning, in this case, it is a punctured

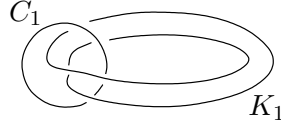


FIGURE 7. The only Seifert fibered reflective augmented link with one knot strand component.

non-orientable surface in the solid torus  $M - C_1$ . These were classified by Tsau [27], and have boundary of the form  $P \cap \partial\mathcal{N}(C_1) = \alpha = q\mu + (2k)\lambda$ , where  $\mu$  is a meridian of the solid torus,  $\lambda$  is a longitude,  $k \geq 1$ , and  $q$  is an odd integer.

In our case, we know which boundary slopes  $\alpha$  can occur, because of the existence of the involution  $\sigma$ . In particular, since  $\partial D_1$  intersects  $P$  exactly twice, by Lemma 2.2,  $k = 1$ . Then by untwisting, we may assume  $q = 1$ , and so  $\alpha$  is the slope  $\mu + 2\lambda$ , and  $P$  is a Möbius band in the solid torus  $M - C_1$ .

The knot strands are embedded in the Möbius band  $P$  and nontrivial. Thus they must be parallel to  $\partial P$ . If there is just one knot strand, the link  $L$  is as in Figure 7. More precisely, it has complement homeomorphic to the complement of the link of Figure 7 in  $S^3$ .  $\square$

We obtain the following immediate consequence of this result.

**Corollary 3.9.** *Let  $K$  be a knot in  $S^3$  which has a diagram whose augmentation is a Seifert fibered reduced generalized augmented link. Then  $K$  is a  $(2, q)$  torus knot.*

*Proof.* Let  $L$  denote the augmentation of  $K$ . First we show  $S^3 - L$  is irreducible. The manifold  $S^3 - L$  is homeomorphic to  $(S^3 - K) - (\cup C_i)$ , where  $C_i$  are crossing circles encircling generalized twist regions. Let  $S$  be a sphere in  $S^3 - L$ , and assume it does not bound a ball in  $S^3 - L$ . Then the image of  $S$  under the homeomorphism is a sphere in  $(S^3 - K) - (\cup C_i)$  which does not bound a ball. Since  $S^3 - K$  is irreducible,  $S$  must bound a ball  $B$  in  $S^3 - K$ . Hence some  $C_i$  lies in  $B$ . But  $C_i$  is unknotted, hence bounds a disk in  $B$ . This contradicts property (1) of the definition of reduced, Definition 1.1.

By Theorem 3.8, the diagram  $D$  can have only one generalized twist region. The augmentation is the link shown in Figure 7, since there is just one component  $K$ . Thus when we twist to obtain  $S^3 - K$ , we remove  $C_1$  from the diagram and add an even number of crossings at the twist region it bounds. This is a  $(2, q)$  torus knot.  $\square$

#### 4. ESSENTIAL TORI AND AUGMENTED LINKS

In this section, we consider reduced generalized augmented links  $L$  such that  $M - L$  is toroidal. We show that the torus decomposition of  $M - L$  satisfies some nice properties.

Recall that by work of Jaco and Shalen [15] and Johannson [16], every irreducible 3-manifold  $N$  with (possibly empty) torus boundary contains a pairwise disjoint collection of embedded essential tori  $\mathbb{T}$ , unique up to isotopy, such that the closure of a component of  $N - \mathbb{T}$  is either atoroidal or Seifert fibered. If  $N$  admits an orientation reversing involution  $\sigma$ , then by the equivariant torus theorem, first proved by Holzmann [14], each incompressible torus in  $N$  is isotopic to one which is preserved by  $\sigma$  or taken off itself. Then  $\sigma$  applied to  $\mathbb{T}$  gives a new torus decomposition of  $N$ , which by uniqueness must agree with  $\mathbb{T}$ . Thus the closure of each component of  $N - \mathbb{T}$  is either fixed by  $\sigma$ , or taken off itself. This is the equivariant torus decomposition of Bonahon and Siebenmann [8]. We refer to this equivariant torus decomposition as the JSJ decomposition.

**Theorem 4.1.** *Let  $L$  be a reduced generalized augmented link in  $M$ , such that  $M - L$  is irreducible. Then there exists a collection  $\mathcal{T}$  of tori, incompressible in  $M - L$ , such that if  $U$  is a component of  $M - \mathcal{T}$ ,  $U$  satisfies one of the following:*

- $U$  does not meet any crossing circles of  $L$ .
- $U$  meets exactly one crossing circle  $C_i$ ,  $U$  is homeomorphic to  $T^2 \times I$ , and  $C_i$  is isotopic to a simple closed curve on  $\partial U$ .
- $U - (L \cap U)$  is the complement of a reduced generalized augmented link.

The collection of tori  $\mathcal{T}$  may be larger than the minimal collection of the JSJ decomposition. However, we find  $\mathcal{T}$  by adding incompressible tori to the JSJ decomposition.

**Lemma 4.2.** *Let  $L$  be a reduced generalized augmented link in  $M$  such that  $M - L$  is irreducible. Let  $\mathbb{T}$  denote the tori of the JSJ decomposition for  $M - L$ . Suppose there is a component  $U$  of  $M - \mathbb{T}$  that contains a crossing circle  $C_j$  and an embedded annulus  $A$  with one boundary component on some  $T_0 \subset \partial U$  and one on  $C_j$ . Let  $T_j$  be the torus obtained by taking the boundary of a small regular neighborhood of the union of  $T_0$ ,  $A$ , and  $C_j$ . Then  $T_j$  is incompressible in  $M - L$ .*

The collection  $\mathcal{T}$  of Theorem 4.1 will be obtained by adding to  $\mathbb{T}$  any  $T_j$  of Lemma 4.2 which are not isotopic to tori in  $\mathbb{T}$ .

Before proving the lemma, we illustrate by example a situation in which the lemma will apply. Consider the link in Figure 8. The heavy line in that figure shows the location of an incompressible torus  $T_0$ . When we cut along  $T_0$ , we obtain a hyperbolic generalized augmented link on the outside, homeomorphic to the complement of the Borromean rings. On the inside,  $C_1$  and  $T_0$  bound an annulus. Take the boundary of a regular neighborhood of the union of this annulus with  $T_0$  and  $C_1$ , and we obtain an incompressible torus  $T_1$  as in Lemma 4.2. Cutting along  $T_1$ , we split the inside into two components, one homeomorphic to  $(T^2 \times I) - C_1$ , and the other (in this example) another copy of the Borromean rings complement.

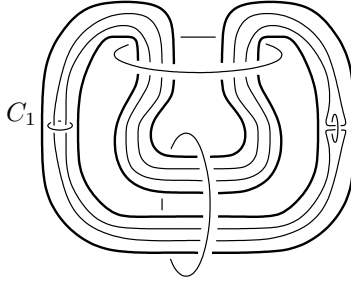


FIGURE 8.  $C_1$  and the torus denoted by the thick line bound an annulus.

Recall that we are interested in links obtained by twisting generalized augmented links. When we do twisting along  $C_1$  in the above example, the component  $(T^2 \times I) - C_1$  becomes the manifold  $T^2 \times I$ . Thus the two incompressible tori  $T_0$  and  $T_1$  become isotopic to each other after twisting. By the results in [20], sufficiently high twisting along the remaining crossing circles in each gives a manifold which remains hyperbolic. Thus the torus decomposition of the twisted link contains just one of  $T_0$  and  $T_1$ . We will see this in Theorem 4.7.

*Proof of Lemma 4.2.* Suppose by way of contradiction that  $T_j$  is compressible. A compressing disk can be isotoped to lie in  $U - (L \cap U)$ , else we obtain a compressing disk for the torus  $T_0$  in  $\mathbb{T}$ , which is impossible.

Surger along a compressing disk for  $T_j$  in  $U - (L \cap U)$  to obtain a sphere  $S$  embedded in  $U - (L \cap U)$ . Note that  $U - (L \cap U)$  is irreducible, since  $M - L$  is irreducible and  $T_0$  is incompressible. Thus  $S$  bounds a ball  $B$  in  $U - (L \cap U)$ . Now,  $B$  cannot be on the side of  $S$  containing  $A$  since this side contains boundary components  $T_0$  and  $C_j$  of  $U - (L \cap U)$ . Thus  $T_j$  bounds a solid torus  $V$  in  $U - (L \cap U)$ . Recall that  $T_j$  was formed by taking the boundary of a regular neighborhood of the union of  $T_0$ , an annulus  $A$ , and  $C_j$  in  $U$ . Since  $T_j$  bounds a solid torus  $V$  in  $U - (L \cap U)$ , it must be the case that  $U - (L \cap U)$  is homeomorphic to  $V$  union a regular neighborhood of the annulus  $A$ . This has just two boundary components:  $T_0$  and  $C_j$ . We claim this is impossible.

Since  $U$  contains  $C_j$ , the surface  $P$  of Proposition 2.1 meets  $U$ , and the torus boundary components of  $U$  are preserved by  $\sigma$ . Hence  $U - (\cup C_i)$  is preserved by  $\sigma$ , where the union is over  $C_i$  in  $U$ . The annulus  $A$  has one boundary component,  $A_1$ , say, on  $\partial\mathcal{N}(C_j)$ , taken by  $\sigma$  to  $-A_1$ . Hence it is isotopic to an annulus which meets  $P$  in two arcs and is preserved under  $\sigma$ . Since tori  $T_0$  and  $\partial\mathcal{N}(C_j)$  are also preserved under  $\sigma$ , the solid torus  $V$  is preserved under  $\sigma$ . Then  $V \cap P$  must be an annulus, a Möbius band, or two meridional disks in  $V$ .

We form  $U - (L \cap U)$  by attaching a thickened annulus to  $V$ . This thickened annulus is attached along some slope  $\mu$  on  $\partial V$ . Since  $A$  is taken to itself



with reversed orientation by  $\sigma$ , the slope  $\mu$  must be taken to  $-\mu$  by  $\sigma$ . There are very few possibilities for  $\mu$ .

In case  $V \cap P$  is an annulus or Möbius band,  $\mu$  must bound a disk in  $V$ . Attaching an annulus to  $V$  along two meridians gives a manifold with compressible boundary, but neither  $C_j$  nor  $T_0$  is compressible.

Thus  $V \cap P$  consists of two meridional disks, and  $\mu$  must be some longitude of  $\partial V$ . When we attach a thickened annulus to longitude slopes, the resulting manifold is homeomorphic to  $T^2 \times I$ . Then  $T_0$  is parallel to  $C_j$ , contradicting the fact that  $T_0$  is essential in  $M - L$ .  $\square$

**Lemma 4.3.** *Let  $C_i$  and  $C_j$ ,  $i \neq j$ , be crossing circles that satisfy the hypotheses of Lemma 4.2. Then the tori  $T_i$  and  $T_j$  are not isotopic.*

*Proof.*  $T_i$  is formed by taking the boundary of a regular neighborhood of the union of an incompressible torus, an annulus with boundary on the torus and on  $C_i$ , and  $C_i$ . Thus there is an annulus  $A_i$  with boundary on  $T_i$  and on  $C_i$ . Similarly for  $T_j$  there is an annulus  $A_j$  with boundary on  $T_j$  and on  $C_j$ . If  $T_i$  and  $T_j$  are isotopic, then the annulus  $A_j$  may be isotoped to have boundary on  $C_j$  and on  $T_i$ . Then we form a new annulus between  $C_i$  and  $C_j$  by taking the union of  $A_i$ ,  $A_j$ , and an annulus running along  $T_i$  between these two. This contradicts the fact that  $L$  is reduced.  $\square$

Form the collection  $\mathcal{T}$  of Theorem 4.1 by adding to  $\mathbb{T}$  all the tori  $T_j$  of Lemma 4.2 which are not isotopic to a torus in  $\mathbb{T}$ . By Lemma 4.3, these are not isotopic to each other. The closure of each component of  $(M - L) - \mathcal{T}$  is still either atoroidal or Seifert fibered, but  $\mathcal{T}$  may no longer be the minimal such collection. By construction, components of  $M - \mathcal{T}$  either do not contain crossing circles of  $L$ ; contain a single crossing circle  $C_j$  sandwiched between incompressible tori  $T_0$  and  $T_j$  of Lemma 4.2, in which case the second possibility of Theorem 4.1 holds; or  $U$  contains crossing circles of  $L$ , but none of these bound annuli with boundary on  $\partial U$ . For the proof of Theorem 4.1, we need to examine these components, and show that any such  $U - (L \cap U)$  is homeomorphic to the complement of a reduced generalized augmented link.

Generalized augmented links are defined to lie in a 3-manifold admitting a reflection through a surface  $S$ . First, we define a manifold  $N$  which will play the role of thus underlying 3-manifold.

Let  $U$  be a component of  $M - \mathcal{T}$  which contains at least one crossing circle  $C_i$ , but does not contain an embedded annulus with boundary on  $\partial U$  and on  $C_i$ , for any  $C_i \subset U$ . Consider  $D_i \cap U$ . In the manifold  $M$ ,  $D_i$  may intersect essential tori of  $\mathcal{T}$ . Then  $D_i$  will meet boundary components of  $U$ . Let  $\ell_{i_1}, \dots, \ell_{i_k}$  be the boundary components of that component of  $D_i \cap U$  which meets  $C_i$ . Replace  $U$  by the manifold  $N_i$  obtained by Dehn filling  $V$  along the slopes  $\ell_{i_j}$ ,  $j = 1, \dots, k$ . Let  $\hat{K}_{i_1}, \dots, \hat{K}_{i_k}$  denote the solid tori attached in the Dehn filling.

Do this Dehn filling for each  $C_i$  contained in  $U$ . We obtain a new manifold  $N$ . Note  $N$  is well-defined because the  $D_i$  are disjoint; thus if  $T_k$  is met by

$D_i$  and  $D_j$ , then  $D_i \cap T_k$  and  $D_j \cap T_k$  must give the same slope, so the Dehn fillings along that slope are the same. Similarly,  $D_i$  might meet  $T_k$  several times, but again along the same slope.

Now, let  $L_U$  be the link in  $N$  consisting of components  $C_i \cap U$  and  $K_j \cap U$ , as well as the cores of each distinct solid torus in the set  $\{\hat{K}_k\}$ . We will abuse notation slightly and continue to refer to these cores of solid tori by  $\hat{K}_k$ . The following is immediate.

**Lemma 4.4.** *For  $N$  the manifold and  $L_U$  the link in  $N$  constructed as above,  $N - L_U$  is homeomorphic to  $U - (L \cap U)$ .  $\square$*

We claim that  $L_U$  is a reduced generalized augmented link in  $N$ , with components  $C_i \cap U$  taking the role of the crossing circles, and components  $K_j \cap U$  and  $\hat{K}_k$  taking the role of the knot strands.

By assumption, there is at least one  $C_i$ , bounding  $D_i$ , meeting at least one  $K_i$  or  $\hat{K}_i$ . So the link  $L_U$  contains at least the minimal number of necessary link components to be a generalized augmented link.

**Lemma 4.5.** *The manifold  $N$  admits an involution through a surface  $S \subset N$ , a link  $K$  is contained in a neighborhood of  $S$ , has diagram  $D$  and maximal twist region selection such that when we encircle generalized twist regions of  $D$  by crossing circles and untwist, the result is a link isotopic to  $L_U$ . That is,  $L_U$  is a generalized augmented link, given by augmenting a diagram of  $K$  in  $N$ .*

*Proof.* The involution  $\sigma$  of Proposition 2.1 preserves  $U - (\cup C_i)$ , where the union is over crossing circles  $C_i$  in  $U$ . It has fixed point set  $U \cap P$ , and components  $K_j$  in  $L_U$  are embedded in  $P$ . We show that the involution  $\sigma$  extends to the solid tori  $\mathcal{N}(\hat{K}_k)$ , and that the cores  $\hat{K}_k$  are embedded in the surface  $P$ .

Recall that  $\mathcal{N}(\hat{K}_k)$  has boundary which is an incompressible torus  $T_k$  in  $M - L$ , and  $T_k$  is preserved by  $\sigma$ . Moreover, some  $D_j$  meets  $T_k$  in a meridian of  $\mathcal{N}(\hat{K}_k)$ . The slope  $\partial D_j \cap T_k$  cannot bound a disk in  $M - L$  by incompressibility of  $T_k$ . Since it does bound a disk in  $D_j$ , this disk must be punctured by some component  $K_i$  in  $M - L$ . Therefore the slope  $\partial D_j \cap T_k$  must be taken by  $\sigma$  to  $-\partial D_j \cap T_k$ . This means a meridian of the solid torus  $\mathcal{N}(\hat{K}_k)$  is inverted by the involution  $\sigma$ . Since the boundary is preserved by  $\sigma$ , it follows that the involution extends to give an involution of the solid torus  $\mathcal{N}(\hat{K}_k)$ . Also,  $P \cap \mathcal{N}(\hat{K}_k)$  must be a longitude of  $\partial \mathcal{N}(\hat{K}_k)$ , in the sense that it intersects the meridian exactly once. Therefore the core,  $\hat{K}_k$ , is embedded in  $P$ .

Now note that  $N$  and  $L_U$  satisfy the conclusion of Proposition 2.1. Since  $\sigma$  acts on  $N$  as an extension of the involution of Proposition 2.1 acting on  $M - (\cup C_i)$ , we know that  $P$  meets the  $C_i$  in  $N$  in the same way it meets the  $C_i$  in  $M$ . Namely,  $P$  either meets  $\mathcal{N}(C_i)$  in two meridian components, as on the left in Figure 5, or in a single component as on the right in Figure 5.

Form the surface  $S$  by taking  $\hat{S}$  to be  $P$  outside of a neighborhood of those crossing disks that meet half-twists. Inside a neighborhood of a half-twist,  $P$  will appear as on the right of Figure 5. The surface  $\hat{S}$ , however, should run straight through the crossing circle, meeting  $\mathcal{N}(C_i)$  in two meridians, as on the left of Figure 5. Note there is a reflection  $\tau$  in  $\hat{S}$  which still preserves  $N - (\cup C_i)$ , although it reverses crossings at half-twists. Moreover, the reflection  $\tau$  preserves a meridian of each crossing circle, hence extends to a reflection  $\tau$  of  $N$  through the surface  $S$ , where  $S$  is obtained from  $\hat{S}$  by capping off boundary components on the  $\mathcal{N}(C_i)$  by disks.

Notice that the strands  $K_i$  and  $\hat{K}_k$  lie in a neighborhood of  $S$ , as do the crossing circles  $C_i$ . Hence when we twist along crossing circles, we form a link  $K$  which still lies in a neighborhood of  $S$ . Moreover, we may project  $K$  to  $S$  such that the twists obtained by twisting along the  $C_i$  form distinct generalized twist regions, and so that  $L_U$  is an augmentation of a diagram of  $K$ .  $\square$

We now need to show that  $L_U$  is reduced in  $N$ . To do so, we find disjoint embedded disks bounded by the crossing circles  $C_j$  in  $N$ , and show these are minimal in the sense of part (1) of Definition 1.1.

**Lemma 4.6.** *Let  $C_{i_1}, \dots, C_{i_k}$  be the crossing circles of  $L_U$ . Each bounds a disk  $D'_{i_j}$  in  $N$  such that the collection  $\{D'_{i_j}\}$  is embedded in  $N$ , the involution  $\sigma$  restricts to an orientation reversing involution of each  $D'_{i_j}$ , and the disks meet  $K_i$  and  $\hat{K}_i$  in  $m_i$  points, where  $m_i \geq 2$  and  $m_i$  is minimal over all disks in  $M$  bounded by  $C_i$ .*

*Proof.* Let  $C_{i_1}, \dots, C_{i_k}$  be the crossing circles in  $U$ . Each  $C_{i_j}$  bounds a disk  $D_{i_j}$  in  $M$ . By construction of  $N$ , the collection  $\{D_{i_1}, \dots, D_{i_k}\}$  extends to an embedded collection of disks  $\{D'_{i_1}, \dots, D'_{i_k}\}$  in  $N$ , such that  $\sigma$  restricts to an orientation reversing involution of each  $D'_{i_j}$ .

If the  $m_i$  is not minimal, then we can find a collection of disjoint, embedded punctured disks which meet  $\partial U$  fewer times. Replace the collection  $\{D'_{i_1}, \dots, D'_{i_k}\}$  with this new collection. Each must meet  $K_i$  and  $\hat{K}_i$  in at least 2 points, for otherwise we would have an annulus between some  $C_j$  and a component  $K_i$  or  $\hat{K}_i$ . The first cannot happen by definition of a reduced generalized augmented link in  $M$ . The second cannot happen by assumption:  $U$  is assumed to be a component of  $M - \mathcal{T}$  which does not contain an embedded annulus with boundary components on  $C_i$  and on  $\partial U$ , and  $\partial \mathcal{N} \hat{K}_i$  is a component of  $\partial U$ .  $\square$

We are now ready to complete the proof of Theorem 4.1.

*Proof of Theorem 4.1.* Let  $\mathcal{T}$  be the collection of tori described after the proof of Lemma 4.3. Let  $U$  be a component of  $M - \mathcal{T}$ . If  $U$  does not contain any crossing circle  $C_i$ , then we are done. If  $U$  contains a crossing circle  $C_i$  and an embedded annulus with boundary  $C_i$  and boundary on  $\partial U$ , then by

construction of  $\mathcal{T}$ ,  $U$  is homeomorphic to  $T^2 \times I$ ,  $C_i$  is the only crossing circle contained in  $U$ , and the curve  $C_i$  is boundary parallel in  $U$ . This is the second case of the theorem.

So assume  $U$  contains a crossing circle, but does not contain any embedded annuli with boundary on  $\partial U$  and on  $C_i$ . Then by Lemma 4.4,  $U - (L \cap U)$  is homeomorphic to the manifold  $N - L_U$ , which, by Lemma 4.5 is a generalized augmented link complement. By Lemma 4.6,  $L_U$  satisfies condition (1) of the definition of reduced, Definition 1.1. It satisfies condition (2) as well, since any annulus embedded in  $N - L_U$  with boundary components on  $C_i$  and  $C_j$  is embedded in  $M - L$  with boundary components on  $C_i$  and  $C_j$ . Since  $L$  is reduced, no such annulus exists. The link  $L_U$  satisfies condition (3) of Definition 1.1 by assumption, given the definition of  $\mathcal{T}$ .  $\square$

**Theorem 4.7.** *Let  $L$  be a reduced generalized augmented link in  $M$  with  $M - L$  irreducible, and let  $\mathcal{T}$  be the tori of Theorem 4.1. Let  $K$  be the link formed by twisting along all the crossing circles of  $L$ , subject to the restriction that if  $C_i$  is contained in a component of  $M - \mathcal{T}$  which is not homeomorphic to  $T^2 \times I$ , then at least 6 half-twists are inserted when we twist along  $C_i$ . Then there is a torus decomposition of  $M - K$  for which components of the decomposition*

- *are either atoroidal or Seifert fibered,*
- *are in one-to-one correspondence with the components of  $M - \mathcal{T}$  which are not homeomorphic to  $T^2 \times I$ ,*
- *and have the same geometric type (hyperbolic or Seifert fibered) as the corresponding component of  $M - \mathcal{T}$ .*

**Remark 4.8.** The decomposition of Theorem 4.7 may not be the JSJ decomposition. In particular, there may be two Seifert fibered components which our decomposition separates, but which are considered as one in the minimal JSJ decomposition. We will see in Section 6 that this does not happen when  $L$  comes from the augmentation of a knot in  $S^3$ , but it could happen more generally.

*Proof.* Let  $U$  be a component of  $M - \mathcal{T}$ . If  $U$  contains no crossing circles, then twisting does not affect  $U$  and so the result holds.

Similarly, if  $U$  is homeomorphic to  $T^2 \times I$ , and contains just a single boundary parallel  $C_i$ , then twisting yields a manifold homeomorphic to  $T^2 \times I$ , which will not be a component of a torus decomposition.

If  $U$  contains a crossing circle, but is not  $T^2 \times I$ , then  $U - (L \cap U)$  is either hyperbolic or Seifert fibered.

If it is hyperbolic, by [20, Proposition 3.5], the slope of the twisting on a horoball neighborhood of a crossing circle  $C_i$  has length at least  $\sqrt{(1/4) + c_i^2}$ , where  $c_i$  is the number of half-twists inserted. Since  $c_i \geq 6$ , this length is greater than 6, hence by the 6-Theorem [2, 17], the result of Dehn filling is hyperbolic.

If  $U - (L \cap U)$  is Seifert fibered, Theorem 3.8 implies  $U - (L \cap U)$  is homeomorphic to the complement of parallel strands embedded in an annulus or Möbius band in a solid torus. After twisting, we obtain a  $(k, k)$  torus link, or a  $(2k, pk)$  torus link, where  $k$  is the number of knot strands of  $L_U$ . These torus links are still Seifert fibered.  $\square$

## 5. REDUCING KNOT DIAGRAMS

We wish to apply the previous results to as many knots and links as possible.

In this section, we prove that all knots in  $S^3$  admit a diagram such that the augmentation is reduced, as in Definition 1.1. We say the diagram  $D$  of a link  $K$  is *twist reduced* if there exists a maximal twist region selection such that the corresponding augmentation of  $K$  gives a reduced generalized augmented link.

**Theorem 5.1.** *Let  $K$  be a knot in  $S^3$  with diagram  $D$  and a maximal twist region selection. Then there exists a twist reduced diagram  $D'$  for  $K$ .*

We will find the diagram of Theorem 5.1 by forming the augmentation of the given diagram, and then removing unnecessary crossing circles and extracting unnecessary knot strands from crossing disks. When we do twisting on remaining crossing circles, projecting twists to the projection plane in  $S^3$  in the usual way, we will obtain the desired diagram of the theorem.

**Definition 5.2.** A *standard diagram* of a generalized augmented link is a diagram such that all knot strands lie on the projection plane except at half-twists, which are contained in a neighborhood of the corresponding crossing circle. Crossing circles are perpendicular to the projection plane, and crossing disks project to straight lines running directly under the crossing circles of the diagram. For example, the portions of the diagrams in Figures 2(b), 3, and 4 are standard.

The next few lemmas ensure part (1) of Definition 1.1 will hold.

**Lemma 5.3.** *Let  $L$  be a generalized augmented link in  $S^3$  with standard diagram. Suppose there exists a disk  $E$  embedded in  $S^3$ , with boundary some crossing circle  $C_i$ , disjoint from the other crossing circles, and suppose that  $E$  meets the knot strands fewer times than does  $D_i$ . Then there exists such a disk  $F$  such that in addition,  $F \cap (\cup \text{int}(D_j)) = \emptyset$ .*

*Proof.* Suppose  $E$  meets the interior of some  $D_j$ . We may assume the intersection is transverse and consists of simple closed curve components. There is some innermost disk  $\hat{E}$  on  $E$  whose boundary is a curve  $\gamma$  on  $D_j$ . Consider the disk  $G$  constructed by taking the disk  $D_j$  outside  $\gamma$ , replacing  $D_j$  inside  $\gamma$  by  $\hat{E}$ . Push  $G$  off  $D_j$  slightly, so  $G$  and  $D_j$  do not intersect.

Suppose first that  $G$  meets the knot strands fewer times than does  $D_j$ . Then replace  $E$  by  $G$ , replacing  $C_i$  by  $C_j$ . This disk  $G$  has fewer intersections with  $\cup \text{int}(D_k)$  than does  $E$ .

Suppose instead  $G$  meets the knot strands the same number of times as does  $D_j$ . Replace  $E$  by replacing  $\hat{E} \subset E$  with the portion of  $D_j$  bounded by  $\gamma$ , and push off  $D_j$ . We have decreased the number of intersections of  $E$  with  $D_j$  without increasing the number of intersections of  $E$  with knot strands.

In either case, we have a new disk which meets the interiors of the  $D_k$  fewer times. Repeat a finite number of times, and we obtain a disk  $F$  as in the statement of the lemma.  $\square$

**Lemma 5.4.** *Let  $L$  be a generalized augmented link in  $S^3$  with standard diagram. Suppose there is a disk  $E$  embedded in  $S^3$  such that  $E$  has boundary some crossing circle  $C_i$ , is disjoint from the other crossing circles, and  $E \cap (\cup \text{int}(D_k))$  is empty. Then there exists such a disk  $F$  such that in addition,  $F$  intersects the projection plane in a single arc  $\gamma_1$ , the intersections of  $F$  with knot strands all lie on  $\gamma_1$ , and the number of intersections of  $F$  with knot strands is at most the number of intersections of  $E$  with the knot strands.*

*Proof.* First, we show we may assume that  $E$  does not meet the diagram of  $L$  in the neighborhood of any  $D_j$  containing a half-twist. That is, we show we may assume  $E$  does not meet any half-twists of the diagram. Assume  $E$  does run through a half-twist corresponding to  $D_j$ . The half-twist is contained in a neighborhood of  $D_j$ , which is homeomorphic to  $D_j \times [-1, 1]$ . Without loss of generality, assume the half-twist is contained in  $D_j \times (-1, 0)$ , with  $D_j$  lying at  $D_j \times \{0\}$ . Since  $E$  does not meet  $D_j$ , it must intersect  $\partial(D_j \times (-1, 0))$  in the surfaces  $D_j \times \{-1\}$  or  $\partial D_j \times (-1, 0)$ . If any intersections of  $E$  with  $\partial(D_j \times (-1, 0))$  bound disks in  $\partial(D_j \times (-1, 0)) - L$ , then we may replace  $E$  by replacing corresponding disks in  $E$  with those in  $\partial(D_j \times (-1, 0)) - L$ , and pushing into  $D_j \times (-1, 0)$  slightly. After this replacement, we may assume that any component of  $E \cap \partial(D_j \times (-1, 0))$  lies on  $D_j \times \{-1\}$  and bounds punctures of  $D_j - L$ . Since  $(D_j \times (-1, 0)) - L$  is homeomorphic to  $(D_j - L) \times (-1, 0)$ , we may replace  $E$  by replacing a disk bounded by  $E \cap (D_j \times \{-1\})$  in  $E$  by the corresponding disk in  $D_j$ , and pushing out of  $D_j \times (-1, 0)$  slightly. This will meet the knot strands of  $L$  at most as many times as does  $E$ , and will not intersect the neighborhood of  $D_j$  containing the half-twist.

So assume  $E$  does not meet the diagram of  $L$  in any half-twists. Since outside of half-twists, all knot strands lie on the projection plane, any intersections of  $E$  with the knot strands must lie on the projection plane. It remains to show we can assume there is just one component of intersection of  $E$  with the projection plane.

Let  $\gamma_1$  be the arc of intersection of  $E$  with the projection plane whose endpoints lie on  $C_i$ . Consider  $S^3$  cut along the projection plane. Remove neighborhoods of all the  $D_j$  for  $j \neq i$ , including all half-twists, as well as a neighborhood of a half-twist at  $D_i$ , if applicable, which does not contain  $C_i$ . This gives two balls, with  $\gamma_1$  embedded in the boundary of each ball, and one half of  $C_i$  embedded as an arc in each ball, with endpoints meeting those of

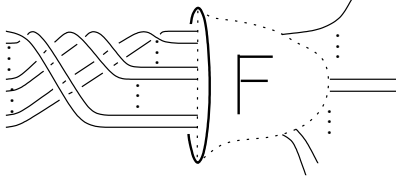


FIGURE 9.

$\gamma_1$ . Note no components of  $L$  intersect the interior of either ball, aside from  $C_i$ . Thus in each ball we may find an embedded disk with boundary running along  $C_i$  and along the arc  $\gamma_1$  which only meets  $L$  in its boundary. These disks glue to give a new disk  $F$  which meets the knot strands exactly where  $\gamma_1$  meets the knot strands, hence meets the knot strands at most as many times as does  $E$ . The disk  $F$  satisfies the conclusions of the lemma.  $\square$

**Lemma 5.5.** *Let  $L$  be a generalized augmented link in  $S^3$  with standard diagram. Suppose there is a disk  $E$  in  $S^3$  with boundary some crossing circle  $C_i$ , disjoint from the other crossing circles, such that  $E$  meets the knot strands fewer times than does  $D_i$ , and  $E \cap (\cup D_j)$  is empty. Then we may isotope  $L$  to a generalized augmented link with standard diagram with the same crossing circles, and the same number of strands running through each crossing circle, except that  $D_i$  meets the knot strands fewer times.*

*Proof.* Consider the intersection of the sphere  $E \cup D_i$  with the projection plane. By Lemma 5.4, we may assume this intersection is a single simple closed curve  $\gamma$ , with one arc of the curve running along the intersection of  $D_i$  with the projection plane, and the other arc meeting the knot strands exactly in the intersections of  $E$  with the knot strands.

If  $C_i$  bounds a half-twist, we may isotope the diagram such that all the crossings of the half-twist lie outside of the sphere  $E \cup D_i$ , as in Figure 9. In any case, the portion of the diagram containing the sphere consists of  $m_i$  strands running parallel, embedded on the projection plane, entering  $D_i$ , and  $n_i < m_i$  parallel strands, embedded on the projection plane, exiting  $E$ .

Isotope to move the ball bounded by  $E \cup D_i$  to the opposite side of  $C_i$ , as in Figure 10. Notice that we have a new diagram, with the same crossing circles as before, and the same numbers of knot strands running through each crossing circle, except there are now  $n_i$  strands running through  $C_i$ .

If  $C_i$  does not bound a half-twist, then this is a standard diagram of a generalized augmented link, and the lemma is proved. If  $C_i$  does bound a half-twist, then this is no longer a standard diagram, since the half-twist associated with  $C_i$  no longer remains in a neighborhood of  $C_i$  after the isotopy. In this case, we perform a flype on the region of the diagram bounded by  $\gamma$ . That is, we rotate  $180^\circ$  in the opposite direction of the half twist. This cancels the crossing of the “old” half-twist associated with  $C_i$ ,



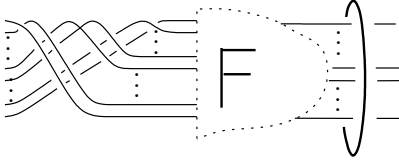


FIGURE 10.

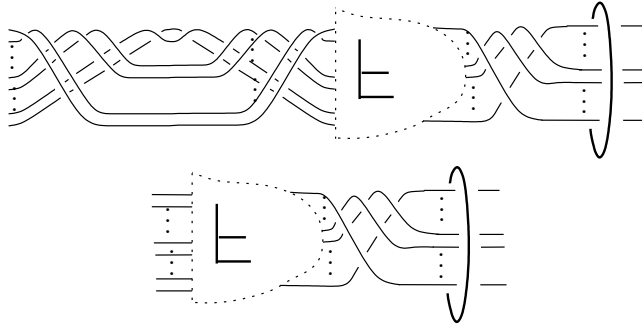


FIGURE 11.

and adds a half-twist in the immediate neighborhood of  $C_i$  after the isotopy, as in Figure 11.

Finally, notice that the flype does not add any crossings, aside from those of the “new” half-twist adjacent to  $C_i$ . For within the region bounded by  $\gamma$ , knot strands on the projection plane are taken back to the projection plane under the flype. Crossing circles are taken to crossing circles. Half-twists rotate  $180^\circ$  to become identical half-twists. Outside the region bounded by  $\gamma$ , the diagram does not change at all, except to shift the half-twist from one side of  $\gamma$  to the other. Thus in this case we have established the lemma.  $\square$

**Lemma 5.6.** *Given a standard diagram of a generalized augmented link  $L$  in  $S^3$ , there exists a new diagram that is also standard, but for which  $D_i$  meets the knot strands in the minimal number of points.*

*Proof.* If the original diagram does not satisfy this property, then combining Lemmas 5.3 and 5.5 we obtain a new diagram for which the number of strands running through a single crossing circle has been reduced. Notice that there are only a finite number of crossing circles, and a finite number of strands running through each crossing circle. Thus we need only repeat a finite number of times, and we are left with a diagram for which  $D_i$  meets the knot strands in the minimal number of points.  $\square$

The next lemmas will give property (2) of Definition 1.1.

**Lemma 5.7.** *Let  $L$  be a generalized augmented link in  $S^3$  with standard diagram. Suppose there is an incompressible annulus  $A$  embedded in  $L$  with boundary  $C_i$  and  $C_j$ , for some  $i \neq j$ . Then  $A$  is isotopic to an annulus which does not meet  $D_i$  or  $D_j$ , and intersects  $\cup \text{int}(D_k)$  in a (possibly empty) collection of core curves of the annulus.*

That is, any such annulus must either completely run through a twist region, or completely miss the twist region.

*Proof.* First, apply Lemma 5.6 to the diagram of  $L$ . We may assume  $L$  has standard diagram for which each  $D_i$  meets the knot strands in the minimal number of points.

Consider the intersection of  $A$  with some  $D_k$ . This is a collection of closed curves. Suppose one of them, say  $\alpha$ , bounds a disk  $D$  in  $A$ . Then the disk obtained by taking  $D_k$  outside  $\alpha$  and  $D$  inside  $\alpha$  must meet the knot strands the same number of times as  $D_k$ , since  $D_k$  meets the knot strands the minimal number of times. Thus  $\alpha$  must bound a disk in  $D_k$ . Hence we can isotope off, reducing the number of intersections.

Now suppose  $A$  intersects  $D_i$ . Then by the preceding paragraph, any intersection must be an essential curve  $\alpha$  in  $A$ . Thus we may isotope  $C_i$  along  $A$  to this curve of intersection, and then push off slightly, reducing the number of intersections of  $A$  with  $D_i$ . Repeat, until there are no intersections with  $D_i$ . Similarly for  $D_j$ .  $\square$

**Lemma 5.8.** *Let  $L$  be a generalized augmented link in  $S^3$ . Suppose there is an embedded annulus  $A$  in  $S^3 - L$  with boundary  $C_i$  and  $C_j$ ,  $i \neq j$ . Then there is a generalized augmented link  $L'$  in  $S^3$  such that first,  $S^3 - L$  is homeomorphic to  $S^3 - L'$ , second, there is a one-to-one correspondence between crossing circles and knot strands of  $L$  and  $L'$ , third, twisting disks of a standard diagram of  $L$  meet the knot strands the same number of times as the corresponding twisting disks in a standard diagram of  $L'$ , but the crossing circle corresponding to  $C_i$  does not meet the knot strands of  $L'$  in a half-twist.*

*Proof.* Start with a standard diagram of  $L$ , and consider  $C_i$  and  $C_j$ . If one of  $C_i$  and  $C_j$  does not encircle a half-twist, then  $L$ , possibly with  $C_i$  and  $C_j$  switched, satisfies the conclusions of the lemma.

So suppose both  $C_i$  and  $C_j$  encircle half-twists. Isotope  $A$  as in Lemma 5.7. Then  $A \cup D_i \cup D_j$  is a sphere  $S$  in  $S^3$ . We may isotope  $C_i$  and  $C_j$  such that the half-twists bounded by these crossing circles are both on the outside of  $S$ .

Now perform a flype on the inside of  $S$  in the direction opposite the half-twist at  $C_i$ . We wish to analyze what happens to the diagram after the flype. First, consider the portions of the diagram inside  $S$ . These are rotated  $180^\circ$ . This rotation takes strands on the projection plane back to the projection plane, and takes crossing circles to crossing circles without

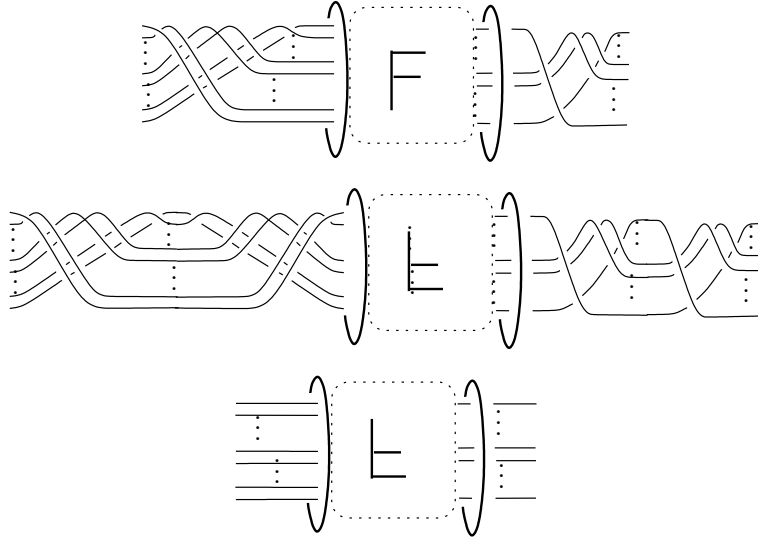


FIGURE 12. Top to bottom: the link  $L$  before the flype. After the flype. The link  $L'$ .

affecting the number of strands running through each crossing circle. Finally, the rotation takes half-twists to identical half-twists, in the same direction.

Outside of  $S$ , the flype does not affect any strands of the diagram, except that it removes the half-twist at  $C_i$  and adds a new half-twist at  $C_j$ . This will either cancel with the half-twist already at  $C_j$ , or it will add to it, putting a full-twist at  $C_j$ . In the case the flype leaves a full-twist at  $C_j$ , we replace the flyped diagram with one in which the full twist at  $C_j$  has been removed, replaced by parallel strands with no crossings at  $C_j$ , as in Figure 12. This link is the link  $L'$ . The complement of  $L'$  is homeomorphic to the complement of  $L$ .

To finish the proof of the lemma, we need to show that the given diagram of  $L'$  is the standard diagram of a generalized augmented link.

If no portion of the diagram of  $L$  on the outside of  $S$  crosses over a portion of the diagram of  $L$  on the inside of  $S$ , then this will still be true for  $L'$ , and since both outside and inside of  $S$  in  $L'$  we have portions of diagrams of generalized augmented links, their union must be the diagram of a generalized augmented link.

So suppose some portion of the diagram of  $L$  outside of  $S$  crosses over some portion diagram inside  $S$ . These will form crossings of the original standard diagram of  $L$ . All crossings are associated with half-twists. Since the crossings of the half-twists associated with  $C_i$  and  $C_j$  lie outside of  $S$ , and do not meet  $S$ , these crossings must be associated to some  $C_k$  with  $k \neq i, j$ . Then we may assume that  $S$  intersects the corresponding twisting disk  $D_k$ , or we may isotope  $S$  out of a neighborhood of  $D_i$  as in Lemma 5.4, reducing the number of crossings of the outside of  $S$  with the inside of  $S$ .

So assume  $S$  intersects  $D_k$ . By Lemma 5.7,  $D_k$  must meet the annulus  $A$  in its non-trivial core curve. Then the portion of the diagram of  $L$  inside  $S$  at this intersection will be a half-twist. When we perform the flype, the half-twist becomes an identical half-twist in the same direction. Thus in a neighborhood of  $D_k$  inside of  $S$ , the diagram remains unchanged after the flype. Since the diagram does not change outside of  $S$ , a neighborhood of  $D_k$  will still be a half-twist after the flype. Thus in all cases, the diagram of  $L'$  is the standard diagram of a generalized augmented link.  $\square$

*Proof of Theorem 5.1.* Let  $K$  be a knot in  $S^3$  with a maximal twist region selection, and let  $L$  the corresponding generalized augmented link with standard diagram. If  $L$  is not reduced, then it fails one of properties (1), (2), or (3) of Definition 1.1. Since our knot is embedded in  $S^3$ , which has empty boundary, condition (3) will not hold, so we need only show (1) and (2).

Suppose that property (1) fails. Then by Lemma 5.6, we may replace our diagram by a new standard diagram in which  $D_i$  meets the knot strands in the minimal number of points. Note that *a priori*, there may now be a twisting disk  $D_i$  that meets the knot strands in  $m_i$  points, where  $m_i \leq 1$ . In this case, simply remove  $C_i$  from the diagram, since twisting along  $D_i$  when  $m_i \leq 1$  gives a manifold homeomorphic to that obtained by performing meridional Dehn filling on  $C_i$ .

Suppose property (2) fails. Then there is an annulus embedded in  $S^3 - L$  with boundary on some  $C_i$  and  $C_j$ . By Lemma 5.8, we may replace the diagram of  $L$  with a standard diagram  $D'$  of a generalized augmented link  $L'$ , without increasing the numbers of crossing circles or strands running through crossing circles, such that now the crossing circle corresponding to  $C_i$  doesn't encircle a half-twist in the diagram  $D'$ . Consider the image of the annulus in  $S^3 - L'$  under the homeomorphism of  $S^3 - L$  and  $S^3 - L'$ . This is an annulus embedded in  $S^3 - L$  with one boundary component on  $C_i$ . We may isotope  $C_i$  along this annulus to be parallel to the other boundary component,  $C_j$ . The result has diagram  $D''$  identical to  $D'$ , except that  $C_i$  has been removed and a crossing circle parallel to  $C_j$  has been added.

Now, to consider the knot  $K$ , we perform twisting on  $C_i$  and on  $C_j$ . However, note that both add full-twists to the same generalized twist region. Hence we may remove  $C_i$  from the diagram, and adjust the amount of twisting at  $C_j$  to obtain the same link  $K$ .

Perform Dehn filling along the crossing circles, adding the appropriate number of twists so that the result has complement homeomorphic to  $S^3 - K$ . This has a diagram given by replacing the crossing circles  $C_k$  in the diagram with a number of full-twists. The diagram will be reduced, by construction. Moreover, since knots are determined by their complements [13], the diagram is of a knot isotopic to  $K$ .  $\square$

**Remark 5.9.** Three remarks on the above proof.

First, note that when we reduced the diagram, we only removed crossing circles from the unreduced augmented link, never added. Thus the resulting

diagram will have at most as many generalized twist regions as the original. This is useful for Theorem 7.3 below.

Second, what would be more useful would be if the reduced diagram had at least as many half twists per generalized twist region as the original, since many of our results in this paper require a knot with a high number of half twists per twist region. However, this may not be the case if property (2) of Definition 1.1 does not hold for the original diagram. In that case, we must concatenate two generalized twist regions into a single one. This may cancel half twists in opposite directions, reducing the total number of half twists in the new generalized twist region.

Finally, notice that the above proof used the fact that  $K$  was a knot only in the very last step. To show the same result for links in  $S^3$ , one needs to show that an isotopy of the generalized augmented link followed by the appropriate Dehn filling is equivalent to an isotopy of the original link. We believe this is true (and possibly even known), but we have not worked out the details here.

## 6. APPLICATIONS TO KNOTS

In this section, we apply the results of the previous sections to give geometric information on knots in  $S^3$ . Throughout, we will let  $K$  be a knot in  $S^3$ , and let  $D$  be a twist reduced diagram of  $K$ . By Theorem 5.1, we may assume any knot admits such a diagram.

**6.1. Torus decomposition and geometric type.** Given a knot  $K$  in  $S^3$ , with twist reduced diagram  $D$ , form an augmented link  $L$  by adding crossing circles to the diagram in a maximal twist region selection. We have restrictions on essential tori in  $S^3 - L$ .

**Lemma 6.1.** *Let  $T$  be an essential torus in  $S^3 - L$ . Then  $T$  bounds a solid torus  $V$  such that*

- (1)  $V$  is invariant under the involution  $\sigma$ ,
- (2)  $V$  contains the link component  $K$ ,
- (3)  $V \cap P$  is nonempty, with  $P \cap \partial V$  containing no meridians of  $V$ .

*Proof.* Because  $T$  is essential, it must intersect the surface  $P$ . By the invariant torus theorem [14], we may assume  $T$  is invariant under  $\sigma$ . By the solid torus theorem,  $T$  bounds a solid torus  $V$  in  $S^3$ , which must also intersect  $P$ . Because  $\sigma$  fixes  $P \cap V$  pointwise,  $\sigma$  must preserve  $V$ . This gives the first item of the lemma.

Suppose  $V$  does not contain  $K$ . At least one crossing circle must be inside  $V$ , else a meridian of  $V$  is a compressing disk of  $T$ , contradicting incompressibility of  $T$ . Let  $C_i \subset V$ . The circle  $C_i$  bounds  $D_i$ . If  $D_i$  does not intersect  $\partial V$ , then  $D_i$  cannot intersect  $K$ , contradicting the definition of a generalized augmented link. Thus  $D_i$  must intersect  $\partial V$ .

We may assume all such intersections are nontrivial curves on  $\partial V$ , else we may replace disk portions of  $D_i$  with disks of  $\partial V$ . This gives us a new

$D_i$ , still invariant under  $\sigma$  (since  $V$  is invariant under  $\sigma$ ), with required properties of the definition of a generalized augmented link. So we assume all intersections of  $D_i$  with  $\partial V$  are nontrivial.

Consider a curve of  $D_i \cap \partial V$  that is innermost in  $D_i$ . This bounds a disk  $E$  on  $D_i$ . The disk  $E$  cannot lie in  $V$ , or it would be a compressing disk. Thus it must lie outside  $V$ . But then  $V$  is unknotted in  $S^3$ . Therefore replace  $V$  with the exterior of  $V$  in  $S^3$ . This is a solid torus that contains  $K$ . Thus we have the second item of the lemma.

Finally, suppose  $\partial V \cap P$  consists of meridians of  $\partial V$ , i.e., curves that bound disks in  $V$ . Then since  $V$  is fixed by  $\sigma$ ,  $V \cap P$  must consist of two meridional disks. The knot  $K$  is contained in  $V$ , and  $K$  is contained in  $P$  by Proposition 2.1. Since the  $C_i$  are contained in a neighborhood of  $K$ , the link  $L$  must lie in a neighborhood of a meridional disk of  $V$ . But then  $V$  is compressible. Contradiction.  $\square$

Lemma 6.1 allows us to classify Seifert fibered components of the torus decomposition of  $S^3 - L$ . We have the following.

**Proposition 6.2.** *Let  $K$  be a knot in  $S^3$  with a twist reduced diagram. Let  $L$  be a corresponding augmentation. Let  $\mathcal{T}$  be the tori in the torus decomposition of Theorem 4.1. Then the tori of  $\mathcal{T}$  are nested, bounding solid tori  $V_1 \supset V_2 \supset \cdots \supset V_k \supset K$ , and if  $U$  is a component of  $S^3 - \mathcal{T}$  which is not homeomorphic to  $T^2 \times I$ , such that  $U - (L \cap U)$  is Seifert fibered, then  $U$  is the outermost component of the decomposition, and  $U - (L \cap U)$  is homeomorphic to the manifold of Figure 7.*

*Proof.* Since each torus of  $\mathcal{T}$  is essential in  $S^3 - L$ , by Lemma 6.1 each bounds a solid torus invariant under  $\sigma$ , containing  $K$ , with  $V \cap P$  a longitudinal slope. Thus we can arrange the solid tori in order of containment:

$$K \subset V_k \subset \cdots \subset V_1.$$

By Theorem 4.1, for each component  $U$  of  $S^3 - \mathcal{T}$  which is not homeomorphic to  $T^2 \times I$ ,  $U - (L \cap U)$  is a reduced augmented link complement. Thus it has a boundary component corresponding to some  $C_j$ . Then any such component except the outermost must have at least three boundary components: at least one  $C_j$ , some  $V_i$ , and  $V_{i+1}$  (or  $K$ ). Now, if this component is Seifert fibered, Theorem 3.8 tells us that  $V_i$  and  $V_{i+1}$  must be parallel on  $P$ , and therefore  $V_{i+1}$  cannot be a subset of  $V_i$ . This is impossible.

So only the outermost component may be Seifert fibered. In this case, again Theorem 3.8 tells us there is just one  $C_1$  in the component  $(S^3 - L) - V_1$ , and  $(S^3 - V_1) - C_1$  is a solid torus. Since there is only one other link component, namely  $V_1$ , it must be embedded in the Möbius band  $P$ , parallel to the boundary of  $P$ , as in Figure 7.  $\square$

**Corollary 6.3.** *Let  $K$  be a torus knot with a twist reduced diagram  $D$  and a maximal twist region selection in which each twist region admits at least 6 half twists. Then  $K$  is a  $(2, p)$  torus knot, and  $D$  has one twist region.*

*Proof.*  $S^3 - K$  is Seifert fibered. Since  $D$  is twist reduced, adding crossing circles to twists of  $D$  yields a reduced augmented link  $L$  in  $S^3$ . By Theorems 4.1 and 4.7, there is a sublink  $\hat{L}$  of  $L$ , possibly consisting of fewer crossing circles, and a collection of tori  $\hat{\mathbb{T}}$  such that the components of  $(S^3 - \hat{L}) - \hat{\mathbb{T}}$  are reduced augmented links with the same geometric type as those of  $S^3 - K$ . Thus the components must all be Seifert fibered.

By Proposition 6.2, only the outermost component is Seifert fibered, and it is of the form of Figure 7. Thus the collection of tori of  $\hat{\mathbb{T}}$  is empty. When we twist to insert at least six half twists, we obtain a  $(2, p)$  torus knot.  $\square$

**Theorem 6.4.** *Let  $K$  be a knot in  $S^3$  which is toroidal, with a twist-reduced diagram and a maximal twist region selection with at least 6 half-twists in each generalized twist region. Let  $L$  denote the corresponding augmentation. Then there exists a sublink  $\hat{L}$  of  $L$ , possibly containing fewer crossing circles, such that:*

- (1) *The essential tori of the JSJ decomposition of  $S^3 - K$  are in one-to-one correspondence with those of  $S^3 - \hat{L}$ .*
- (2) *Corresponding components of the torus decompositions have the same geometric type, i.e. are hyperbolic or Seifert fibered.*
- (3) *The essential tori of  $S^3 - \hat{L}$  and  $S^3 - K$  form a collection of nested tori, each bounding a solid torus in  $S^3$  which contains  $K$ , and which is fixed under the reflection of  $S^3 - L$ .*

*Proof.* By Theorems 4.1 and 4.7, there is some sublink  $\hat{L}$  of  $L$ , possibly containing fewer crossing circles (i.e. those inside any  $T^2 \times I$  components), and tori  $\hat{\mathbb{T}}$  such that the tori form a torus decomposition of  $S^3 - K$ , and corresponding components of the decompositions of  $S^3 - K$  and  $S^3 - \hat{L}$  have the same geometric type. By Lemma 6.1, item (3) must hold for all essential tori.

All that remains to prove is that the decomposition given by  $\hat{\mathbb{T}}$  is the JSJ decomposition, i.e. it is the unique minimal torus decomposition of  $S^3 - \hat{L}$  and  $S^3 - K$ . If not, then some torus  $T$  of  $\hat{\mathbb{T}}$  separates two Seifert fibered components. But by Proposition 6.2, this is impossible: only the outermost component of  $S^3 - \hat{L}$  can be Seifert fibered. Thus any other components must be hyperbolic, and so  $\hat{\mathbb{T}}$  is the unique torus decomposition of  $S^3 - \hat{L}$ .

Similarly, the only way  $\hat{\mathbb{T}}$  can fail to be the unique torus decomposition of  $S^3 - K$  is if some essential torus  $T$  of  $\hat{\mathbb{T}}$  splits a single Seifert fibered component into two. But again the components of  $S^3 - K$  have the same geometric type as those of  $S^3 - \hat{L}$ , by Theorem 4.7. Hence  $\hat{\mathbb{T}}$  must be the unique torus decomposition of  $S^3 - K$ .  $\square$



## 7. APPLICATION: GROMOV NORMS

In this section, we apply the previous results to bound the Gromov norms of many toroidal knots. The results use heavily the particular torus decompositions developed in previous sections, as well as results on hyperbolic generalized augmented links in [20].

First, we insert a bound on the volume of a hyperbolic augmented link in a solid torus, whose proof follows immediately from [20].

**Proposition 7.1.** *Let  $L$  be a generalized augmented link in a solid torus  $V$ , with  $t_i$  crossing circles. Suppose  $V - L$  is hyperbolic. Then its volume satisfies  $\text{vol}(V - L) \geq 2 v_8 t_i$ , where  $v_8 \approx 3.66386$  is the volume of a hyperbolic regular ideal octahedron.*

*Proof.* By Lemma 4.1 of [20], the volume of a hyperbolic generalized augmented link in  $S^3$  with  $t$  generalized twist regions is at least  $2 v_8(t - 1)$ . We may use this result to bound volumes of hyperbolic generalized augmented links in a solid torus as follows. Embed the solid torus as one of the solid tori of a standard genus one Heegaard splitting of  $S^3$ . Let  $C$  be the core of the other solid torus. Then the embedding gives a generalized augmented link in  $S^3$  where the curve  $C$  is an additional crossing circle component. Thus the volume of a hyperbolic augmented link in a solid torus is at least  $2 v_8(t_i + 1 - 1) = 2 v_8 t_i$ , where  $t_i$  denotes the number of crossing circles.  $\square$

Alternately, Proposition 7.1 may be proved by cutting  $V - L$  along  $P$  and crossing disks, and using this to determine the number of vertices and edges of a one-skeleton as in the proof of Lemma 4.1 of [20].

**Theorem 7.2.** *Let  $K$  be a knot with twist reduced diagram with  $t$  generalized twist regions. Let  $L$  be the corresponding generalized augmented link and let  $t_0$  denote the number of components of the torus decomposition of the form  $(T^2 \times I) - C_i$ . Then the Gromov norm of  $S^3 - L$  satisfies*

$$\|[S^3 - L]\| \geq 2 v_3 v_8(t - t_0 - 1),$$

where  $v_3 \approx 1.0149$  is the volume of a regular ideal hyperbolic tetrahedron, and  $v_8 \approx 3.66386$  is the volume of a regular ideal hyperbolic octahedron.

*Proof.* By Proposition 6.2 and Theorem 4.1,  $S^3 - L$  admits a torus decomposition such that each piece of the decomposition is either a link in  $T^2 \times I$ , or a generalized augmented link in a solid torus. Moreover, the only Seifert fibered pieces each meet just one crossing circle component, either as the outermost component of the decomposition or as a link in  $T^2 \times I$ . Since the Gromov norm is  $v_3$  times the sum of the volumes of the hyperbolic pieces [26], we use the results of Proposition 7.1.

Now, in case the outermost piece is hyperbolic, it is an augmented link in  $S^3$  and so its volume is at least  $2 v_8(t_1 - 1)$ , where  $t_1$  denotes the number of crossing circles in the outermost piece. Then the Gromov norm is at least

$$\|[S^3 - L]\| \geq v_3 \left( 2 v_8(t_1 - 1) + \sum 2 v_8 t_i \right) = 2 v_3 v_8 (t - t_0 - 1),$$

where the sum in the center is over all components of the torus decomposition which are homeomorphic to hyperbolic generalized augmented links in a solid torus.

If the outermost piece is hyperbolic, then it contributes nothing to the Gromov norm. By Proposition 6.2, it contains just one crossing circle, and so the Gromov norm is at least

$$\|[S^3 - L]\| \geq v_3 \sum 2 v_8 t_i = 2 v_3 v_8 (t - t_0 - 1),$$

where the final  $-1$  corresponds to the single crossing circle in the outermost component of the torus decomposition.  $\square$

For knots:

**Theorem 7.3.** *Let  $K$  be a knot in  $S^3$  which is toroidal, with a twist-reduced diagram at least 7 half-twists in each generalized twist region. Let  $L$  denote the corresponding augmentation, and let  $\hat{L}$  denote the sublink of Theorem 6.4. Let  $t$  denote the number of crossing circles of  $\hat{L}$ . Then the Gromov norm of  $S^3 - K$  satisfies*

$$\|[S^3 - K]\| \geq 0.65721 (t - 1).$$

*Proof.* Since the Gromov norm is  $v_3$  times the sum of volumes of hyperbolic pieces of the torus decomposition, we bound volumes of the hyperbolic pieces. Each is obtained by Dehn filling a generalized augmented link. Moreover, by [20, Proposition 3.5], the length of the slope of Dehn filling is at least  $\sqrt{(1/4) + c_i^2}$ , where  $c_i$  is the number of half-twists. Thus if there are at least 7 half-twists in each twist region, then the Dehn filling slopes have length at least  $\sqrt{49.25} \approx 7.0178 > 2\pi$ .

Now apply [12, Theorem 1.1]. This theorem bounds the volume under Dehn filling in terms of the length of the filling slope and the volume of the unfilled manifold. In particular, for twisting a hyperbolic augmented link with  $t_i$  crossing circles in a solid torus, by Proposition 7.1 we have

$$\text{volume after twisting} \geq \left(1 - \left(\frac{2\pi}{\sqrt{49.25}}\right)^2\right)^{3/2} (2 v_8 t_i) > 0.64756 t_i.$$

Only the outermost component of the torus decomposition may be Seifert fibered. If it is not, the volume of the outermost piece is at least  $0.64756 (t_1 - 1)$ , where  $t_1$  is the number of crossing circles, by [20, Theorem 4.2]. Since  $\hat{L}$  is obtained from  $L$  by removing all crossing circles which lie in components  $(T^2 \times I) - C_j$  of the torus decomposition of  $S^3 - L$  (Theorem 4.1), the total number of crossing circles in  $\hat{L}$  is  $t_1 + \sum t_i$ , where the sum is over all pieces homeomorphic to hyperbolic generalized augmented links in solid tori. Thus the Gromov norm satisfies

$$\|[S^3 - K]\| \geq v_3 \left(0.64756 (t_1 - 1) + \sum 0.64756 t_i\right) = 0.65721 \dots (t - 1).$$

If the outer most component is Seifert fibered, then the outermost component contains a single crossing circle component, by Proposition 6.2, and so the Gromov norm satisfies

$$\| [S^3 - K] \| \geq v_3 \sum 0.64756 t_i = 0.65721 \dots (t - 1).$$

□

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